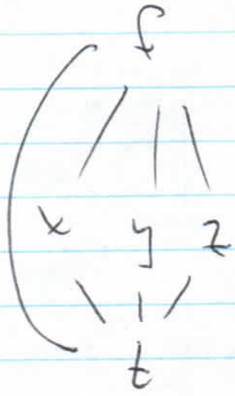
Substantial Derivative

Consider the function $f(x,y,z,t)$ where $x(t)$ $y(t)$ $z(t)$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial t}$$

$$= \underline{V}\cdot\nabla f + \frac{\partial f}{\partial t}$$



so if $f = \rho\eta$

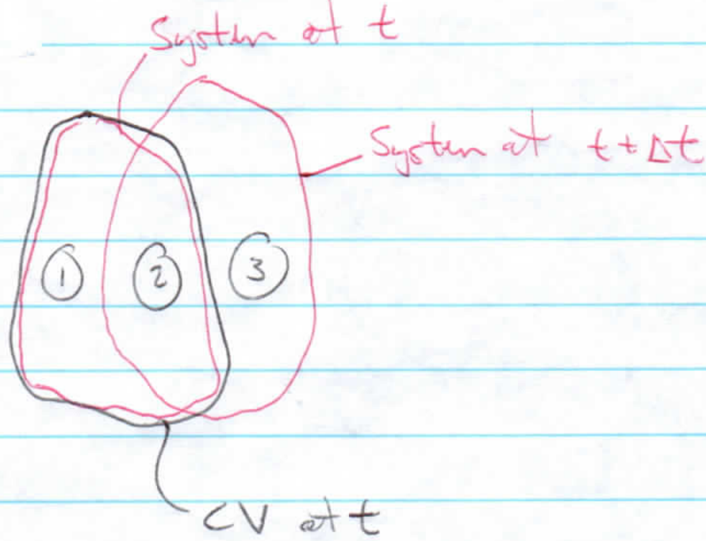
$\rho \sim$ mass density m/vol

$\eta \sim$ stuff/mass

$f \sim$ stuff/vol

$$\frac{d}{dt}(\rho\eta) = \underline{V}\cdot\nabla(\rho\eta) + \frac{\partial}{\partial t}(\rho\eta)$$

Reynolds Transport Thm



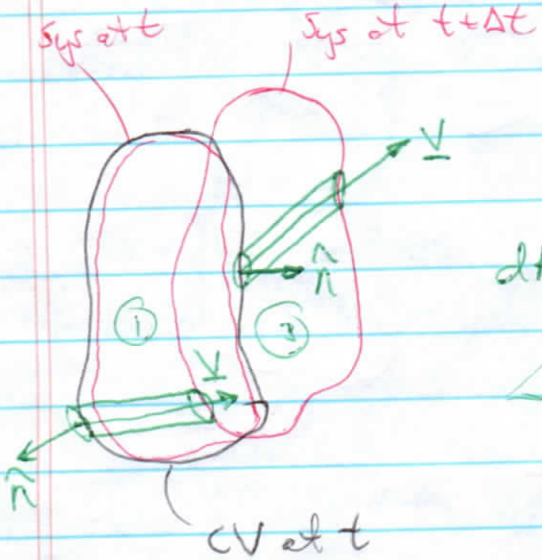
Look at $\frac{D N_{sys}}{Dt} = \lim_{\Delta t \to 0} \frac{N_{sys}(t+\Delta t) - N_{sys}(t)}{\Delta t}$

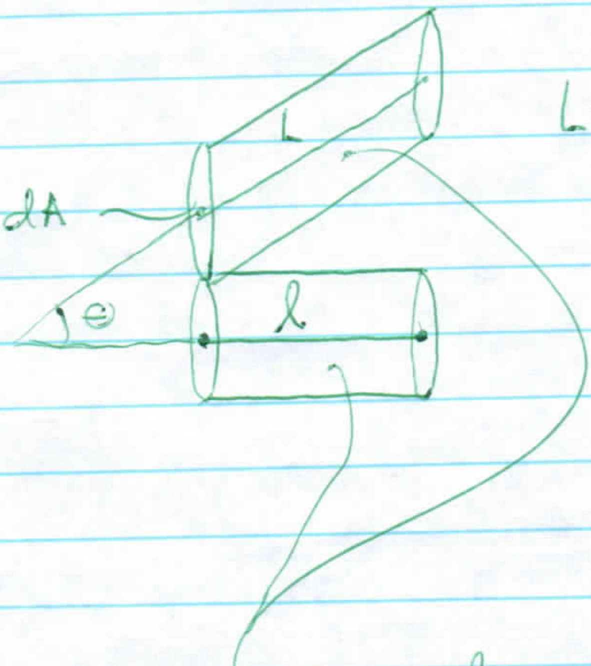
$$= \lim \frac{[N_3(t+\Delta t) + N_2(t+\Delta t)] - [N_2(t) + N_1(t)]}{\Delta t}$$

$$= \lim \frac{[N_2(t+\Delta t) + \overbrace{N_1(t+\Delta t)}^{*}] - [N_2(t) + N_1(t)] + N_3(t+\Delta t) - \overbrace{N_1(t+\Delta t)}^{*}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{N_{cv}(t+\Delta t) - N_{cv}(t)}{\Delta t} + \lim_{\Delta t \to 0} \frac{N_3(t+\Delta t) - N_1(t+\Delta t)}{\Delta t}$$

$$= \frac{d}{dt} \iiint_{CV} \rho\, \eta \, d\forall + \underbrace{\lim_{\Delta t \to 0} \frac{N_3(t+\Delta t) - N_1(t+\Delta t)}{\Delta t}}_{\text{lets look at this}}$$





$L = |\underline{V}|\,\Delta t$

$$
\begin{aligned}
dV &= dA \cdot l \\
&= dA \cdot L\cos\theta \\
&= dA\, |\underline{V}|\,\Delta t \cos\theta \\
&= dA\, \underline{V}\cdot\hat{n}\, \Delta t
\end{aligned}
$$

So $$\frac{N_3(t+\Delta t) - N_1(t+\Delta t)}{\Delta t} = \frac{\displaystyle\iint_{C.Surf} \rho\eta\, \underline{V}\cdot\hat{n}\, dA\, \Delta t}{\Delta t}$$

$$\therefore \lim_{\Delta t \to 0} \frac{N_3(t+\Delta t) - N_1(t+\Delta t)}{\Delta t} = \iint_{C.S.} \rho\eta\, \underline{V}\cdot\hat{n}\, dA$$

$\underline{V}$ stuff/boundary

We are finally left with

Version 1

$$\frac{DN_{sys}}{Dt} = \frac{d}{dt}\iiint_{CV} s\eta \, d\forall + \iint_{C.S.} s\eta \, \underline{V}_{s/b} \circ \hat{n} \, dA$$

Reynolds Transport Thm.

$\underline{V}$ = velocity of stuff relative to the boundary $= \underline{V}_{s/b}$

Can also write this as (using Leibnitz Thm)

$$\frac{d}{dt}\iiint_{CV} s\eta \, d\forall = \iiint_{CV} \frac{\partial}{\partial t}(s\eta) \, d\forall + \iint_{C.S.} s\eta \, V_{b/o} \circ \hat{n} \, dA$$

in which case RTT becomes

Version 2

$$\frac{D}{Dt} N_{sys} = \iiint_{CV} \frac{\partial}{\partial t}(s\eta) \, d\forall + \iint_{CS} s\eta \left(\underline{V}_{s/b} + \underline{V}_{b/o}\right) \circ \hat{n} \, dA$$

G.D.T. $\iint \underline{F} \circ \hat{n}\, d\sigma = \iiint \nabla \circ \underline{F}\, dV$

G.R.

$$\frac{\partial}{\partial t} \iiint_R \rho\, dV + \iint_S \rho \underline{V} \circ \hat{n}\, d\sigma = \iiint_R \nabla \circ (\rho \underline{V})\, dV$$

let $\underline{F} = \rho \underline{V}$

Vect. Identity

$\nabla \circ (\rho \underline{V}) =$

$\rho \nabla \circ \underline{V} + \nabla \rho \circ \underline{V}$

$\int$ To Differential Form

Conservation of mass...

$$\frac{D\,Mass}{Dt} = 0 = \iiint \frac{\partial \rho}{\partial t}\, d\forall + \iint \rho\, \underset{s/0}{\underline{V}} \circ \hat{n}\, d\sigma$$

$\iint \underline{F} \circ \hat{n}\, d\sigma$

$\parallel$ G. Div. Thm. $= \iiint \nabla \circ \underline{F}\, d\forall$

$$\iiint \nabla \circ (\rho \underline{V})\, d\forall$$

$$0 = \iiint_{CV} \left[\frac{\partial \rho}{\partial t} + \nabla \circ (\rho \underline{V})\right] d\forall$$

but the CV was arbitrary so the integrand must be 0 any/every where

$$\therefore \boxed{\frac{\partial \rho}{\partial t} + \nabla \circ (\rho \underline{V}) = 0}$$ Cons. of mass

$$= \nabla \rho \circ \underline{V} + \rho \nabla \circ \underline{V}$$

$$\underbrace{\frac{\partial \rho}{\partial t} + \nabla \rho \circ \underline{V}} + \rho \nabla \circ \underline{V} = 0$$

$$\boxed{\frac{D\rho}{Dt} + \rho \nabla \circ \underline{V} = 0}$$ CONS. OF MASS

For soilds + fluids $\rho \approx \text{¢}$

reduces to $\nabla \circ \underline{V} = 0$

To look at Mom + E + S first later rewrite RTT's ~~left hand side~~ RHS

$$\frac{DN_{sys}}{Dt} = \iiint_{CV} \frac{\partial \rho\eta}{\partial t}\, d\forall + \iint_{CS} \rho\eta \underline{V}_{s/o} \circ \hat{n}\, d\sigma$$

↓ G. D. Thm.

$$= \iiint_{CV} \left[\frac{\partial \rho\eta}{\partial t} + \nabla \circ (\rho\eta \underline{V}) \right] d\forall$$

$$= \iiint \left[\frac{\partial \rho\eta}{\partial t} + \rho\eta \nabla \circ \underline{V} + (\nabla \rho\eta) \circ \underline{V} \right] d\forall$$

$$= \iiint \left[\frac{D \rho\eta}{Dt} + \rho\eta \nabla \circ \underline{V} \right] d\forall$$

$$= \iiint \left[\rho \frac{D\eta}{Dt} + \frac{D\rho}{Dt}\eta + \rho\eta \nabla \circ \underline{V} \right] d\forall$$

$$= \iiint \left[\rho \frac{D\eta}{Dt} + \eta \underbrace{\left(\frac{D\rho}{Dt} + \rho \nabla \circ \underline{V} \right)}_{=0 \text{ by } m} \right] d\forall$$

$$\boxed{\frac{DN_{sys}}{Dt} = \iiint_{CV} \rho \frac{D\eta}{Dt}\, d\forall}$$

We will use this for the rest of this on the RHS of x-Mom E

E

$$\frac{DE_{sys}}{Dt} = \underbrace{\iiint \frac{\partial \rho e}{\partial t}\, d\forall + \iint_{cs} \rho e\, \underline{V}_{s/\sigma} \cdot \hat{n}\, d\sigma}$$

$$\overbrace{-\iint \underline{q} \cdot \hat{n}\, d\sigma - \iint p \underline{V} \cdot \hat{n}\, d\sigma + \iiint U''' d\forall} = \iiint_{cv} \rho \frac{De}{Dt}\, d\forall$$

Turn surface integrals into volume integrals

$$-\iiint \nabla \cdot \underline{q}\, d\forall - \iiint \nabla \cdot (p\underline{V})\, d\forall + \iiint U''' d\forall = \iiint \rho \frac{De}{Dt}\, d\forall$$

or $\iiint [\ \ ]\, d\forall = 0 \Rightarrow [\ \ ] = 0$

so

$$\rho \frac{De}{Dt} + \nabla \cdot (\underline{q} + p\underline{V}) = U'''$$

But $e = u + \frac{V^2}{2} + gz$ so we are going to break this up into mech energy + thermal energy...

$$\rho \frac{D}{Dt} u + \underbrace{\rho \frac{D}{Dt}\left(\frac{V^2}{2} + gz\right)}_{\text{mech}} + \nabla \cdot \underline{q} + \underbrace{\nabla \cdot (p\underline{V})}_{\text{mech}} = U'''$$

So

E thermal $\boxed{\rho \frac{Du}{Dt} + \nabla \cdot \underline{q} = U'''}$

and the rest of is actually Bernoulli's eqn!

Finally we use $du = C_v\, dT$ heat cap. def.

$\underline{q} = -k \nabla T$ Fourier's Law

(5)

$$\rho \frac{D c_p T}{Dt} = \nabla \cdot (k \nabla T) + u'''$$

If $c_p = ¢$ and $k = ¢$ ($c_p \approx c_v = c$ for solids & liquids)

$$\boxed{\rho c \frac{DT}{Dt} = k \nabla^2 T + u'''}$$

Classic heat cond. eqn for (E)

or

$$\boxed{\rho c \left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = k \nabla^2 T + u'''}$$